Appendix 1 Function atan2

When we consider the problem of finding an angle θ for given consistent values of $\sin\theta$ and $\cos\theta$, it is obvious that there exists a unique solution in the region $-180° < \theta \leqq 180°$, or under a module of 360°. There has not, however, been a commonly used means for expressing this solution by a single function. Function atan2 provides such a means in the form of an extension of the function $\tan^{-1}\theta$. The name "atan2" comes from a subroutine name in computer programming languages that gives a unique solution of the arc tangent function.

Function atan2 is a scalar function of two scalar arguments defined by

$$\operatorname{atan2}(a,b) = \arg(b + ja), \tag{A1.1}$$

where a and b are real numbers, j is the imaginary unit, and arg is the argument of a complex number (see figure A1.1). For notational convenience, we define atan2(0,0) to be indeterminate. Several properties and applications of this function are summarized in this appendix.

It is obvious that

$$\theta = \operatorname{atan2}(\sin\theta, \cos\theta) \tag{A1.2}$$

and

$$\theta = \operatorname{atan2}(k\sin\theta, k\cos\theta) \tag{A1.3}$$

for any positive scalar k.

It is also easy to show the following equalities:

$$\operatorname{atan2}(-a,b) = -\operatorname{atan2}(a,b), \tag{A1.4}$$

$$\tfrac{1}{2}\pi \pm \operatorname{atan2}(a,b) = \operatorname{atan2}(b, \mp a), \tag{A1.5}$$

$$\pi \pm \operatorname{atan2}(a,b) = \pm\operatorname{atan2}(-a, -b). \tag{A1.6}$$

For the differentiation of atan2, we have

$$\frac{\partial \operatorname{atan2}(a,b)}{\partial a} = \frac{b}{a^2 + b^2}, \tag{A1.7a}$$

$$\frac{\partial \operatorname{atan2}(a,b)}{\partial b} = \frac{-a}{a^2 + b^2}. \tag{A1.7b}$$

Hence, when a and b are differentiable functions of time t, we have

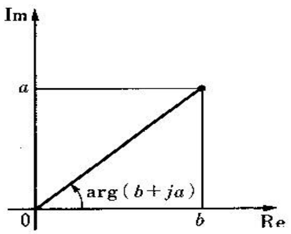

Figure A1.1
Argument of complex number.

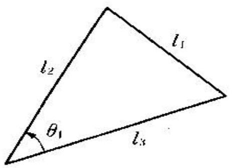

Figure A1.2
Cosine theorem.

$$\frac{d\operatorname{atan2}(a,b)}{dt} = \frac{\dot{a}b - a\dot{b}}{a^2 + b^2}, \tag{A1.8}$$

where $\dot{a} = da/dt$ and $\dot{b} = db/dt$.

As an application, the cosine theorem will be expressed using atan2. For the triangle shown in figure A1.2, assume that $l_1 \geqq 0$, $l_2 > 0$, and $l_3 > 0$. The cosine theorem tells us that

$$l_1{}^2 = l_2{}^2 + l_3{}^2 - 2l_2l_3\cos\theta_1. \tag{A1.9}$$

We let

$$\kappa = \sqrt{(l_1{}^2 + l_2{}^2 + l_3{}^2)^2 - 2(l_1{}^4 + l_2{}^4 + l_3{}^4)}, \tag{A1.10}$$

and then we have

$$2l_2l_3\sin\theta_1 = \pm\kappa.$$

Hence, we obtain from equation A1.3

$$\theta_1 = \pm\operatorname{atan2}(\kappa, l_2{}^2 + l_3{}^2 - l_1{}^2). \tag{A1.11}$$

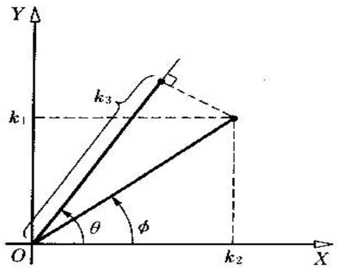

Figure A1.3
Geometric expression of $k_1 \sin\theta + k_2 \cos\theta = k_3$.

Although this expression has been derived under the assumption that $l_2 > 0$ and $l_3 > 0$, it is valid also for the case of $l_2 = 0$ or $l_3 = 0$ in the sense that θ_1 is indeterminate by the notational convention we have made. Note that the cosine theorem does not hold if the condition

$$(l_2 + l_3)^2 \geqq l_1{}^2 \geqq (l_2 - l_3)^2$$

is not satisfied.

As a second application, we derive the solution θ of

$$k_1 \sin\theta + k_2 \cos\theta = k_3, \tag{A1.12}$$

where k_1, k_2, and k_3 are known scalar constants satisfying $k_1{}^2 + k_2{}^2 \geqq k_3{}^2$. Geometrically, as shown in figure A1.3, the problem is to find angle θ between the X axis and a straight line such that k_3 is the distance between the origin and the foot of the normal drawn from point (k_1, k_2) to the straight line. Now we let

$$\phi = \operatorname{atan2}(k_1, k_2). \tag{A1.13}$$

Since

$$k_3 = \sqrt{k_1{}^2 + k_2{}^2} \cos(\theta - \phi), \tag{A1.14}$$

we have

$$\sqrt{k_1{}^2 + k_2{}^2} \sin(\theta - \phi) = \pm\sqrt{k_1{}^2 + k_2{}^2 - k_3{}^2}. \tag{A1.15}$$

Thus, from equations A1.3, A1.14, and A1.15 we obtain

$$\theta - \phi = \pm \operatorname{atan2}(\sqrt{k_1{}^2 + k_2{}^2 - k_3{}^2}, k_3). \tag{A1.16}$$

Therefore, the solution to equation A1.12 is given by

$$\theta = \operatorname{atan2}(k_1, k_2) \pm \operatorname{atan2}(\sqrt{k_1{}^2 + k_2{}^2 - k_3{}^2}, k_3). \tag{A1.17}$$

Function atan2 was first introduced into robotics by R. P. Paul in *Robot Manipulators: Mathematics, Programming, and Control* (MIT Press, 1981).